Abstract

We study stochastic differential games in a bounded domain D in $\mathbb{R}^d$ with a smooth boundary ∂D. Let U_i be a compact metric space which is the control set of the player i, $i = 1, 2$. Let $V_i = \mathcal{P}(U_i)$, the space of probability measures on U_i. We consider a dynamic system evolving in $\overline{D} = D \cup \partial D$ according to the following stochastic differential equation

$$\left.\begin{aligned} dX(t) &= b(X(t), v_1(t), v_2(t))\, dt + \sigma(X(t))\, dW(t) - \gamma(X(t))\, d\xi(t), \\ d\xi(t) &= I\left\{\xi(t) \in \partial D\right\} d\xi(t),\ t \geq 0, \\ X(0) &= x \in D,\ \xi(0) = 0, \end{aligned}\right\}$$

where

$$b : \overline{D} \times V_1 \times V_2 \to \mathbb{R}^d,$$

defined by

$$b(x, v_1, v_2) = \int_{U_2} \int_{U_1} \bar{b}(x, u_1, u_2)\, v_1(du_1)\, v_2(du_2),$$

$$\bar{b} : \overline{D} \times U_1 \times U_2 \to \mathbb{R}^d \quad \text{the drift vector}, \ \sigma : \overline{\mathrm{D}} \to \mathbb{R}^{d \times d} \quad \text{the diffusion coefficient},$$

$$\gamma : \overline{D} \to \overline{D}, \quad \text{a smooth vector field},$$

$W(\cdot)$ is a d-dimensional standard Wiener process, $v_i(\cdot)$ is a V_i-valued nonanticipative process called an admissible strategy of the player i, $i = 1, 2$. The state $X(t)$ is a reflecting diffusion in $\overline{D}$ controlled by the processes $(v_1(\cdot), v_2(\cdot))$. When $X(t)$ hits the boundary ∂D, it reflects inward along the vector field γ. The continuous process $\xi(\cdot)$ increases only when $X(\cdot)$ hits ∂D. Let A_i, $i = 1, 2$, denote the set of all admissible strategies for player i.

In zero-sum game with the above dynamics we consider two payoff criteria: α-discounted and long-run average payoffs. Let

$$\bar{r} : \overline{D} \times U_1 \times U_2 \to \mathbb{R}$$

be the payoff function. When the state is at $x \in \overline{D}$ and player i chooses action $u_i \in U_i, i = 1, 2$, then player 1 receives a payoff $\bar{r}(x, u_1, u_2)$ from player 2. The players have strictly opposite interests; the player 1 wishes to maximize his total income whereas the player 2 wishes to minimize the same. Let $r : \overline{D} \times V_1 \times V_2 \to \mathbb{R}$ be defined as

$$r(x, v_1, v_2) = \int_{U_2} \int_{U_1} \bar{r}(x, u_1, u_2)\, v_1(du_1)\, v_2(du_2)\ .$$

Let $(v_1, v_2) \in A_1 \times A_2$. The α-discounted payoff to player 1 for the initial condition $x \in \overline{D}$ is defined as

$$R_\alpha[v_1, v_2](x) = E\left[\int_0^\infty e^{-\alpha t} r(X(t), v_1(t), v_2(t))\, dt \,\middle|\, X(0) = x\right].$$

A strategy $v_1^* \in A_1$ is said to be *α-discounted optimal for player* 1 for initial condition x if

$$R_\alpha[v_1^*, \tilde{v}_2](x) \geq \inf_{v_2 \in A_2} \sup_{v_1 \in A_1} R_\alpha[v_1, v_2](x),$$

for any $\tilde{v}_2 \in A_2$.
Define the function $\overline{R}_\alpha : \overline{D} \to \mathbb{R}$ by

$$\overline{R}_\alpha(x) = \inf_{v_2 \in A_2} \sup_{v_1 \in A_1} R_\alpha[v_1, v_2](x).$$

The function $\overline{R}_\alpha$ is called the *α-discounted upper value function* of the game.
A strategy $v_2^* \in A_2$ is said to be *α-discounted optimal for player* 2 for initial condition x if

$$R_\alpha[\tilde{v}_1, v_2^*](x) \leq \sup_{v_1 \in A_1} \inf_{v_2 \in A_2} R_\alpha[v_1, v_2](x),$$

for any $\tilde{v}_1 \in A_1$.
Define the function $\underline{R}_\alpha : \overline{D} \to \mathbb{R}$ as follows:

$$\underline{R}_\alpha(x) = \sup_{v_1 \in A_1} \inf_{v_2 \in A_2} R_\alpha[v_1, v_2](x).$$

The function $\underline{R}_\alpha$ is called the *α-discounted lower value function* of the game. If $\overline{R}_\alpha(\cdot) \equiv \underline{R}_\alpha(\cdot)$, then the game is said to admit a value for discounted criterion and the common function is denoted by R_α and is called *α-discounted value function.*
Let $(v_1, v_2) \in A_1 \times A_2$. The long-run average payoff (or ergodic payoff) to the player 1 for the initial condition $x \in \overline{D}$ is defined as

$$L[v_1, v_2](x) = \liminf_{T \to \infty} \frac{1}{T} E\left[\int_0^T r(X(t), v_1(t), v_2(t))\, dt \,\middle|\, X(0) = x\right]$$

The long-run average optimal strategies and the value function are defined in a similar manner as the discounted case.

To establish the existence of a value for the α-discounted payoff we consider the corresponding Isaacs equation given by

$$\left.\begin{aligned}\alpha\phi(x) &= \inf_{v_2\in V_2}\sup_{v_1\in V_1}\left[L^{v_1,v_2}\phi(x)+r(x,v_1,v_2)\right] \quad \text{in } D,\\ &= \sup_{v_1\in V_1}\inf_{v_2\in V_2}\left[L^{v_1,v_2}\phi(x)+r(x,v_1,v_2)\right] \quad \text{in } D,\\ &\frac{\partial\phi}{\partial\gamma} = 0 \quad \text{on } \partial D,\end{aligned}\right\}$$

where

$$L^{v_1,v_2}\phi(x) = \sum_{i=1}^{d} b_i(x,v_1,v_2)\frac{\partial\phi(x)}{\partial x_i} + \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2\phi(x)}{\partial x_i\partial x_j},\ a = \sigma\sigma'.$$

Under certain assumptions on b, σ, r, we show that the $\alpha-$discounted value function is the unique solution in a certain class of functions of the above Isaacs equation We also characterize the optimal strategies for both players using the Isaacs equation. For ergodic payoff criterion we derive analogous results.

For nonzero-sum case we study both $\alpha-$discounted and average payoff criteria. Let $\bar{r}_i$: $\overline{D}\times U_1\times U_2 \to \mathbb{R}$, i = 1, 2, be the payoff function for player i. When the state is $x\in\overline{D}$ and the player i chooses action $u_i\in U_i$, then player i receives a payoff $\bar{r}_i$. Each player tries to maximize his total income.

Let $r_i : \overline{D}\times V_1\times V_2 \to \mathbb{R}$ be defined as

$$r_i(x,v_1,v_2) = \int_{U_2}\int_{U_1}\bar{r}_i(x,u_1,u_2)\,v_1(du_1)\,v_2(du_2).$$

Let $(v_1,v_2)\in A_1\times A_2$. Then $\alpha-$discounted payoff to the player i, $i=1,2$, for the initial condition $x\in\overline{D}$ is defined as

$$R^i_\alpha[v_1,v_2] = E\left[\int_0^\infty e^{-\alpha t}\,r_i(X(t),v_1(t),v_2(t))\,dt\,\Big|\,X(0)=x\right].$$

A pair of admissible strategies $(v_1^*,v_2^*)\in A_1\times A_2$ is said to be an *$\alpha-$discounted (Nash) equilibrium* for the initial condition $x\in\overline{D}$ if

$$\left.\begin{aligned}R^1_\alpha[v_1^*,v_2^*](x) &\geq R^1_\alpha[v_1,v_2^*](x) \quad \text{for all } v_1\in A_1,\\ R^2_\alpha[v_1^*,v_2^*](x) &\geq R^2_\alpha[v_1^*,v_2](x) \quad \text{for all } v_2\in A_2.\end{aligned}\right\}$$

The above inequality shows that if a player unilaterally deviates from a Nash equilibrium, then he is going to be a loser. Let $(v_1,v_2)\in A_1\times A_2$. The long-run average (or ergodic) payoff to the player i for the initial condition $x\in\overline{D}$ is given by

$$L_i[v_1,v_2](x) = \liminf_{T\to\infty}\frac{1}{T}E\left[\int_0^T r_i(X(t),v_1(t),v_2(t))\,dt\,\Big|\,X(0)=x\right].$$

The Nash equlibrium for long-run average payoff is defined analogous to the discounted case. For each criterion we establish the existence of Nash equilibria.
We develop a numerical method to obtain the value function and optimal strategies for the players in the zero-sum game. We prove the convergence of the value function and the optimal strategies of the discretized game to that of the original differential game. We impliment our numerical method to a game of fisheries exploitation.